MATH 2050B 2017-18
Mathematical Analysis I
Make-Up Midterm Solution

(Q1)(i) State the Well-Ordering Principle and the Archimedean Property.

(or an extended vision; you may choose to state the most convenient one for your subsequent use.)

(ii) let $p > 0$, and $x \in \mathbb{R}$. Show that

$$\exists!\ m \in \mathbb{Z},\ \text{such that } x + mp \in (0, p].$$

If $|x - y| < p$ and $x + mp \in (0, p]$, then show that

$$-p < y + mp < 2p.$$

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and of period $p > 0$. Show that f is uniformly continuous on $\mathbb{R}$.

Answer

(i) Well-Ordering Principle

Every nonempty subset of $\mathbb{N}$ has a least element.

That is, for any $\varnothing \neq S \subset \mathbb{N}$, $\exists\ s_* \in S$, such that $s_* \leq s\ \forall\ s \in S$.

(Extended) Archimedean Property

$\forall\ x \in \mathbb{R},\ \exists!\ n \in \mathbb{Z}$, such that $n - 1 \leq x < n$.

(ii) Note since $p \neq 0$, $-\dfrac{x}{p}$ is a well-defined real number.

By (extended) Archimedean Property, $\exists!\ m \in \mathbb{Z}$, such that

$$m - 1 \leq -\frac{x}{p} < m \Longrightarrow 0 < x + mp \leq 1 \Longrightarrow x + mp \in (0, p].$$

If $|x - y| < p$ and $x + mp \in (0, p]$, then $x - p < y < x + p$ and hence

$$-p = 0 - p < x + mp - p < y + mp < x + mp + p \leq p + p = 2p.$$

(iii) Fixed any $\varepsilon > 0$, since f is continuous on $[-p, 2p]$, by uniform continuity theorem,

$$\exists\ \delta' > 0,\ \text{such that } \left|f(s) - f(t)\right| < \varepsilon\ \forall\ s, t \in [-p, 2p] \text{ with } |s - t| < \delta'. \tag{1}$$

Now, take $\delta = \text{Min}\{\delta, p\} > 0$, suppose $x, y \in \mathbb{R}$ with $|x - y| < \delta$,

in particular, $|x - y| < p$, by **(ii)**, $\exists!\ m \in \mathbb{Z}$, such that $x + mp \in (0, p]$ and $y + mp \in (-p, 2p)$.

Note that by f is $p-$periodic, we know $f(x + mp) = f(x)$ and $f(y + mp) = f(x)$.

Also, note that $\left|(x + mp) - (y + mp)\right| = |x - y| < \delta \leq \delta'$.

Then we know by (1)

$$\left|f(x) - f(y)\right| = \left|f(x + mp) - f(y + mp)\right| < \varepsilon.$$

Hence, f is uniformly continuous on $\mathbb{R}$.

(Q2) Let $f : [a, b] \to \mathbb{R}$ be continuous (where $a, b \in \mathbb{R}$ with $a < b$).

Using the Bolzano-Weierstrass Theorem to show that

(i) f is bounded.

(ii) $\exists\, x^*$, such that $f(x) \le f(x^*)\ \forall\, x \in [a, b]$.

Answer

Suppose it were true that f is NOT bounded.

That is, for any $n \in \mathbb{N}$, $\exists\, w_n \in [a, b]$, such that $f(w_n) > n$.

Note $\{w_n\}$ is bounded sequence with $a \le w_n \le b\ \forall\, n \in \mathbb{N}$,

By Bolzano-Weierstrass Theorem,

there is a convergent subsequence $\{w_{n_k}\}$, let it converge to $w \in \mathbb{R}$,

Since $a \le w_{n_k} \le b\ \forall\, k \in \mathbb{N}$, we have $a \le w \le b$, i.e. $w \in [a, b]$,

hence $f(w)$ is a well-defined real number.

Since f is continuous, using sequential criterion of continuous function,

we have $f(w) = \lim_k f(w_{n_k})$.

Fixed any $N \in \mathbb{N}$, we have $f(w_{n_k}) > n_k \ge k \ge N\ \forall\, k \ge N$, it implies $f(w) \ge N$.

That is $f(w) \ge N\ \forall\, N \in \mathbb{N}$, which is a contradiction with Archimedean Property.

Hence, f is bounded.

Since f is bounded, by completeness axiom of $\mathbb{R}$, $s := \text{Sup}\left\{f(x) : x \in [a, b]\right\}$ exists in $\mathbb{R}$.

Hence, $f(x) \le s\ \forall\, x \in [a, b]$. Also, $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in [a, b]$, such that $f(x_n) > s - \frac{1}{n}$.

Note $\{x_n\}$ is bounded sequence with $a \le x_n \le b\ \forall\, n \in \mathbb{N}$,

By Bolzano-Weierstrass Theorem,

there is a convergent subsequence $\{x_{n_k}\}$, let it converge to $x^* \in \mathbb{R}$,

Since $a \le x_{n_k} \le b\ \forall\, k \in \mathbb{N}$, we have $a \le x^* \le b$, i.e. $x^* \in [a, b]$,

hence $f(x^*)$ is a well-defined real number.

Since f is continuous, using sequential criterion of continuous function,

we have $f(x^*) = \lim_k f(x_{n_k})$.

Fixed any $\varepsilon > 0$, by Archimedean Property, $\exists\, N \in \mathbb{N}$, such that $\frac{1}{N} < \varepsilon$.

Then, we have $s \ge f(x_{n_k}) > s - \frac{1}{n_k} \ge s - \frac{1}{k} \ge s - \frac{1}{N} > s - \varepsilon\ \forall\, k \ge N$.

This implies $s \ge f(x^*) \ge s - \varepsilon$, which is $\left|f(x^*) - s\right| \le \varepsilon$ true for all $\varepsilon > 0$.

Therefore, $f(x^*) = s \ge f(x)\ \forall\, x \in [a, b]$.

(Q3) Compute/Guess the limits (in $\mathbb{R} \cup \{-\infty, +\infty\}$):

(i) $\lim_{n\to\infty} r^n$ where $0 < r < 1$.

(ii) $\lim_{x\to-\infty} \frac{1}{2x+99}$.

(iii) $\lim_{x\to 2} \frac{x+1}{x-1}$.

Verify EACH of your assertions by virtue of definition.

Answer

(i) Let $\delta = \frac{1}{r} - 1$, that is $r = \frac{1}{1+\delta}$.

Since $0 < r < 1$, $\delta > 0$.

Using Bernolli's Inequality, $(1+\delta)^n \geq 1 + n\delta \ \forall\, n \in \mathbb{N}$.

That is, $r^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \leq \frac{1}{n\delta} \ \forall\, n \in \mathbb{N}$

Fixed any $\varepsilon > 0$, using Archimedean Property, $\exists\, N \in \mathbb{N}$, such that $N \geq \frac{1}{\delta\varepsilon}$, that is $\frac{1}{N\delta} \leq \varepsilon$.

Then we have

$$\left|r^n - 0\right| = r^n \leq \frac{1}{n\delta} \leq \frac{1}{N\delta} \leq \varepsilon \ \forall\, n \geq N.$$

Therefore, we have $\lim_{n\to\infty} r^n = 0$.

(ii) Fixed any $\varepsilon > 0$, by Archimedean Property, $\exists\, N \in \mathbb{N}$, such that $N > \frac{1}{\varepsilon} + 99$,

it implies $2N > N > \frac{1}{\varepsilon} - 99$, that is $\frac{1}{2N+99} < \varepsilon$.

Now, take $N' = \mathrm{Max}\{N, 50\}$, if $x \in \mathbb{R}$ with $x < -N'$, we have

$x < -50$ and so $\frac{1}{2x+99} < 0$, it means $\left|\frac{1}{2x+99}\right| = \frac{-1}{2x+99}$,

and $x < -N$ and $-(2x+99) > 2N - 99$, it means $\frac{-1}{2x+99} < \frac{1}{2N-99}$.

Now, we have

$$\left|\frac{1}{2x+99} - 0\right| = \frac{-1}{2x+99} < \frac{1}{2N-99} < \varepsilon \ \forall\, x < N'.$$

Therefore, we have $\lim_{x\to-\infty} \frac{1}{2x+99} = 0$.

(iii) Fixed any $\varepsilon > 0$, take $\delta = \mathrm{Min}\left\{\frac{1}{2}, \frac{\varepsilon}{4}\right\} > 0$,

if $x \in \mathbb{R}$ with $0 < |x-2| < \delta$, we have

$$\frac{3}{2} < x < \frac{5}{2} \Longrightarrow \frac{1}{2} < x - 1 < \frac{3}{2} \Longrightarrow 0 < \frac{2}{3} < \frac{2}{x-1} < 2 \Longrightarrow \left|\frac{1}{x-1}\right| < 2$$

if $x \in \mathbb{R}$ with $0 < |x-2| < \delta$, we have

$$\left|\frac{x+1}{x-1} - 3\right| = \left|\frac{x+1-3x+3}{x-1}\right| = \left|\frac{-2x+4}{x-1}\right| = 2|x+2|\left|\frac{1}{x-1}\right| < 2\cdot\delta\cdot 2 \leq \varepsilon.$$

Therefore, we have $\lim_{x\to 2} \frac{x+1}{x-1} = 3$.

(Q4) Let $f : \mathbb{R} \to \mathbb{R}$, and $x_0, l, l' \in \mathbb{R}$ be such that $l \neq l'$ and $\lim_{x \to x_0^-} = l$ and $\lim_{x \to x_0^+} = l'$.

By definition/their negation, show that $f(x)$ does not converge as $x \to x_0$ in $\mathbb{R} \cup \{-\infty, +\infty\}$.

Answer

(Case 1) Suppose it were true that $\lim_{x \to x_0} f(x) = L \in \mathbb{R}$.

Fixed any $\varepsilon > 0$, by all conditions about limit, we have

$\exists\, \delta_1 > 0$, such that for any $x \in \mathbb{R}$ with $0 < x_0 - x < \delta_1$, we have $|f(x) - l| < \frac{\varepsilon}{2}$, and

$\exists\, \delta_2 > 0$, such that for any $x \in \mathbb{R}$ with $0 < x - x_0 < \delta_2$, we have $|f(x) - l'| < \frac{\varepsilon}{2}$, and

$\exists\, \delta_3 > 0$, such that for any $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta_3$, we have $|f(x) - L| < \frac{\varepsilon}{2}$.

Note that $\delta_4 := \text{Min}\{\delta_1, \delta_3\} > 0$, take $x' = x - \frac{\delta_4}{2}$, since $0 < x_0 - x' < \delta_1$ and $0 < |x' - x_0| < \delta_3$,

we have $|l - L| \leq |f(x') - l| + |f(x') - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

That is, $|l - L| \leq \varepsilon$ true for any $\varepsilon > 0$, hence, $|l - L| = 0$ and so $l = L$.

Note that $\delta_5 := \text{Min}\{\delta_2, \delta_3\} > 0$, take $x'' = x + \frac{\delta_5}{2}$, since $0 < x'' - x_0 < \delta_2$ and $0 < |x'' - x_0| < \delta_3$,

we have $|l' - L| \leq |f(x'') - l'| + |f(x'') - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

That is, $|l' - L| \leq \varepsilon$ true for any $\varepsilon > 0$, hence, $|l' - L| = 0$ and so $l' = L$.

This is a contradiction since $l = L = l'$ but $l \neq l'$ by assumption.

Hence, $f(x)$ does not converge as $x \to x_0$ in $\mathbb{R}$.

(Case 2) Suppose it were true that $\lim_{x \to x_0} f(x) = +\infty$.

By all conditions about limit, we have

$\exists\, \delta > 0$, such that for any $x \in \mathbb{R}$ with $0 < x_0 - x < \delta$,

we have $|f(x) - l| < 1$, this implies $f(x) < 1 + l$, and

$\exists\, \delta' > 0$, such that for any $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta'$, we have $f(x) > l + 2$.

Note that $\delta'' := \text{Min}\{\delta, \delta'\} > 0$, take $x' = x - \frac{\delta''}{2}$, since $0 < x_0 - x' < \delta$ and $0 < |x' - x_0| < \delta'$,

we have $f(x') > l + 2 > l + 1 > f(x')$, it implies $0 > 0$ which is a contradiction.

Hence, $f(x)$ does not converge as $x \to x_0$ in $\{+\infty\}$.

(Case 3) Suppose it were true that $\lim_{x \to x_0} f(x) = -\infty$.

By all conditions about limit, we have

$\exists\, \delta > 0$, such that for any $x \in \mathbb{R}$ with $0 < x_0 - x < \delta$,

we have $|f(x) - l| < 1$, this implies $f(x) > l - 1$, and

$\exists\, \delta' > 0$, such that for any $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta'$, we have $f(x) < l - 2$.

Note that $\delta'' := \text{Min}\{\delta, \delta'\} > 0$, take $x' = x - \frac{\delta''}{2}$, since $0 < x_0 - x' < \delta$ and $0 < |x' - x_0| < \delta'$,

we have $f(x') < l - 2 < l - 1 < f(x')$, it implies $0 < 0$ which is a contradiction.

Hence, $f(x)$ does not converge as $x \to x_0$ in $\{-\infty\}$.